- **Answer** 1**:**

  (i) Let G and H be two groups of order p, a prime. Then G, H are cyclic groups of order p. Let $G = \langle a \rangle$ and $H = \langle b \rangle$. Consider the map $\phi : G \longrightarrow H$, and is defined by $a \mapsto b$. One can show that ϕ is an isomorphism. This proves the statement.

  (ii) Consider C_6 and S_3. We know that C_6 is abelian and S_3 non-abelian. So they are non-isomorphic.

- **Answer** 2**:** Let $G = \sqrt[p]{1} = \{z | z^{p^n} = 1\}$, for some n, denote the group of p^nth root of unity under multiplication. Then $G = \langle z \rangle \cong C_{p^n}$, cyclic group of order p^n. Let H be a proper subgroup of G. Let m be the least positive integer such that $z^{p^m} \in H$. We show that H is generated by z^{p^m}.

  Let z^{p^s} be an element of H, and $m \leq s$. Then by division algorithm, $p^s = p^m q + r$, where q and r are integers, $0 \leq r < p^m$. Then $z^{p^s} = z^{p^m q + r}$, so $z^r = z^{p^s - p^m q} \in H$, implies $r = 0$. Therefore, $z^{p^s} = z^{p^m q}$. Since, z^{p^s} be an arbitrary element of H, so z^{p^m} is a generator of H. Hence, it is proved that every proper subgroup of G is cyclic.

- **Answer** 3**:** Let G be a group and $H \neq G$ be a subgroup of finite index. Let $[G : H] = n$. Consider the action of G on G/H by left multiplication. This action induces a homomorphism $\phi : G \longrightarrow \mathrm{Sym}(G/H)$. By First Isomorphism Theorem, we get $G/ker\phi \cong Im(\phi)$. Therefore, $[G : Ker\phi]$ is finite. But Kerϕ is the intersection of conjugates of H. Therefore, H contains Kerϕ, which is a normal subgroup of G of finite index.

- **Answer** 4**:**

  Theorem 0.0.1. *(First Isomorphism Theorem) Let G and H be two groups. Let $\phi : G \mapsto H$ be a group homomorphism. Then $G/Ker\phi \approx Im\phi$.*

The proof of the First Isomorphism Theorem is available in any standard Algebra text book.

- **Answer** 5**:** Let $f : G \longrightarrow H$ be an isomorphism of groups. Then it is clear that f^{-1} is a bijective map. We show that $f^{-1} : H \longrightarrow G$ is a group homomorphism. For $a, b \in H$, $f(f^{-1}(a)f^{-1}(b)) = f(f^{-1}(a))f(f^{-1}(b)) = ab$, then applying both the sides f^{-1}, we get $f^{-1}(a)f^{-1}(b) = f^{-1}(ab)$. Hence, $f^{-1} : H \longrightarrow G$ is a group isomorphism.

- **Answer** 6**:**

  (a) The permutation $(0, 1, 2, 3, 4, 5, 6)$ is not a group automorphism of $\mathbb{Z}/7\mathbb{Z}$ as 0 goes to 1.

  (b) The permutation $(1, 2, 3, 4, 5, 6)$ is not a group automorphism of $\mathbb{Z}/7\mathbb{Z}$. Under the permutation $(1, 2, 3, 4, 5, 6)$, 2 goes to 3, actually 2 should goes to 4.

  (c) The permutation $(1, 2, 4)(3, 6, 5)$ is a group automorphism of $\mathbb{Z}/7\mathbb{Z}$.

  (d) The permutation $(1, 3, 2, 6, 4, 5)$ is a group automorphism of $\mathbb{Z}/7\mathbb{Z}$.

  (e) The permutation $(1, 4, 2)(3, 5, 6)$ is a group automorphism of $\mathbb{Z}/7\mathbb{Z}$.